Differential geometry I $\boxed{\text{Week 8}}$

Last time we stated the following theorem, which provides a useful tool to construct submanifolds of $\mathbb{R}^n$.

<u>Thm</u>: Let $f: U \subseteq \mathbb{R}^m \to \mathbb{R}^n$ be a C^k map of constant rank r, and U be open subset of $\mathbb{R}^m$. Then

A) $\forall q \in f(U) \subseteq \mathbb{R}^n$, the preimage $f^{-1}(q) = \{x \in U : f(x) = q\} \subseteq U$ is a C^k submanifold of <u>codimension</u> r.

B) $\forall p \in U$, there is an open neighborhood $V_p \subseteq U$ of p such that $f(V_p)$ is a C^k submanifold of $\mathbb{R}^n$ of <u>dimension r</u>

$\boxed{\text{(Regular value theorem)}}$

<u>Remark</u>: In general, we do not have $V_p = U$; for example for $f: \mathbb{R} \to \mathbb{R}^2$, $f(x) = (x^2, \sin x)$

we have to restrict to intervals in I on which f doesn't self-intersect.

<u>Proof</u>: The construction of the curvilinear coordinates "flattening out" the submanifolds. Provided by the constant rank theorem, which provides local coordinates in which f looks linear.

<u>Case A</u>: Let $M = f^{-1}(q)$, and $p \in M$.

Since f has constant rank r: By the constant rank theorem, $\exists$ open neighb. V of p in $\mathbb{R}^m$ and W of q in $\mathbb{R}^n$, as well as local coordinates on V, W (given by C^k diffeomorphisms $\phi: V \to V'$, $\psi: W \to W'$)

such that $\psi \circ f \circ \phi^{-1}(x) = (x_1, \dots, x_r, 0, \dots, 0)$

(and $\psi(q) = 0$, $\phi(p) = 0$, $f(V) \subseteq W$).

Since $M \cap V = \{x \in V : f(x) = q\}$,

we have
$$\begin{aligned}
\phi(M \cap V) &= \{y \in V' : f \circ \phi^{-1}(y) = q\} \\
&= \{y \in V' : \psi \circ f \circ \phi^{-1}(y) = 0\} \\
&= \{y \in V' : (y_1, \dots, y_r, 0, \dots, 0) = (0, 0, \dots, 0)\} \\
&= V' \cap \{y : y_1 = \dots = y_r = 0\}
\end{aligned}$$

So M is a C^k submanifold of $\mathbb{R}^n$ of codimension r.

Case B: With ψ, ϕ as above: Let $N = f(V) \subseteq W$.

Then:
$$\begin{aligned}
\psi(N) &= \{z \in W' : z = \psi \circ f(x) \text{ for some } x \in V\} \\
&= \{z \in W' : z = \psi \circ f \circ \phi^{-1}(y) \text{ for some } y \in V'\} \\
&= \{(y_1, \dots, y_r, 0, \dots, 0) \in \mathbb{R}^n : y \in V'\} \cap W' \quad (1)
\end{aligned}$$

If $E = \{y : y_{r+1} = \dots = y_n = 0\} \subseteq \mathbb{R}^n$ and $\pi^m_r : \mathbb{R}^m \to \mathbb{R}^r$ is the projection on the first r coordinates, then

$$\{(y_1, \dots, y_r, 0, \dots, 0) \in \mathbb{R}^n : y \in V'\} = E \cap \underbrace{(\pi^n_r)^{-1}(\pi^m_r(V'))}_{:= V'' \text{ is open}}$$

So from (1):

$$\psi(N) = E \cap V'' \cap W'$$

So if we set $W'' = V'' \cap W'$ (this is open as an intersection of open sets)

~~and we restrict to an open neighborhood Ω of p such that~~

$\cancel{\Omega = f^{-1}(\psi^{-1}(W''))}$: $\psi(N) = E \cap W''$, so N is a submanifold

(of dimension r). $\blacksquare$

The tangent space of a submanifold.

Let $M \subseteq \mathbb{R}^n$ be a C^k $(k \geq 1)$ submanifold of dimension m and $p \in M$.

Definition: A vector v (with base point at p) is a tangent vector of M at p if there exists a curve $\gamma : (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = v$.

The space of all vectors tangent to M at p: $T_p M$ (the tangent space of M at p).

~~Properties: T_pM is a vector space of dimension m~~

Examples: i) If $\gamma : I \to \mathbb{R}^n$ is a regular curve and $M_\gamma = \gamma(I)$, then if $p = \gamma(t_0)$, we have

$$T_p M_\gamma = \{ v \in \mathbb{R}^n : v = \lambda \dot{\gamma}(t_0) \text{ for some } \lambda \in \mathbb{R} \}$$

ii) If $U \subseteq \mathbb{R}^n$ is open (so a submanifold of dimension n)

$\forall p \in U$, $T_p U$ is canonically isomorphic to $\mathbb{R}^n$, since, $\forall v \in \mathbb{R}^n$, $\gamma_v(t) = p + tv$ is a curve in U for t small.

Proposition: If $\dim M = m$, then $\forall p \in M$, T_pM is a vector space of dimension m.

Proof: Let U be a neighborhood of p in $\mathbb{R}^n$ such that, $\exists$ diffeomorphism $\phi : U \to U' \subseteq \mathbb{R}^n$ with

$$\phi(M \cap U) = \underbrace{\{ y \in \mathbb{R}^n : y_{m+1} = \dots = y_n = 0 \}}_{:= E} \cap U'$$

Then:
- For any curve $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = p$, we have (if ε is small enough) $\gamma((-\varepsilon, \varepsilon)) \subseteq M \cap U$ and $\tilde{\gamma}(t) = \phi \circ \gamma(t)$ is inside E for $t \in (-\varepsilon, \varepsilon)$

  $\Rightarrow \dot{\tilde{\gamma}}(0) \in E$ (since E is a vector space)

  But by the composition rule for derivatives:

  $\dot{\tilde{\gamma}}(0) = d\phi_p(\dot{\gamma}(0)) \Longrightarrow d\phi_p(\dot{\gamma}(0)) \in E$

  So $d\phi_p(T_pM) \subseteq E$. Since $d\phi_p$ is injective:

  $\dim T_pM \leq \dim E = m$.

- Conversely, for any $w \in E$: If $\tilde{\gamma}(t) = t \cdot w$ and $\gamma(t) = \phi^{-1}(\gamma(t))$, then $\gamma(t) \in M \cap U$ (for t small) and $\gamma(0) = \phi^{-1}(0) = p$

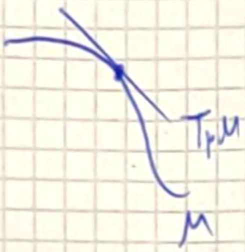
  $\rightarrow \dot{\gamma}(0) \in T_pM$

  But $\dot{\gamma}(0) = d\phi_p^{-1}(w) \Rightarrow d\phi_p^{-1}(E) \subseteq T_pM$

  $\Rightarrow \dim E \leq \dim T_pM \Rightarrow \dim T_pM \geq m$

Proposition: If $f: U \subseteq \mathbb{R}^n \to \mathbb{R}^k$ has constant rank r, then $\forall q \in f(U)$, if $p \in M = f^{-1}(q)$ (this is a submanifold of codimension r)

we have $T_pM = \mathrm{Ker}(df_p)$



Proof: • If $v \in T_pM$: $\exists\, \gamma : (-\varepsilon, \varepsilon) \to M$, with $\gamma(0) = p$ such that $\dot{\gamma}(0) = v$.

Since $\gamma(t) \in f^{-1}(q)$: $f(\gamma(t)) = q = \text{const}$

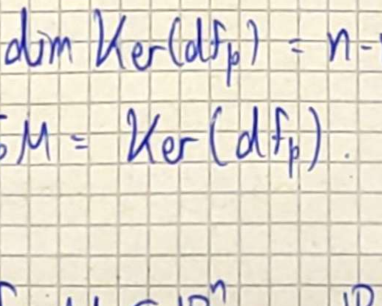
$$\Rightarrow \frac{d}{dt}\left(f(\gamma(t))\right)\Big|_{t=0} = 0$$

$$\Rightarrow df_p(\dot{\gamma}(0)) = 0 \Rightarrow v \in \mathrm{Ker}\, df_p.$$

• So $T_pM \subseteq \mathrm{Ker}(df_p)$.

But $\dim T_pM = n-r$, $\dim \mathrm{Ker}(df_p) = n-r$ (since f has constant rank r), so $T_pM = \mathrm{Ker}(df_p)$. ▢

Important example: If $f: U \subseteq \mathbb{R}^n \to \mathbb{R}$ is smooth, $\nabla f \neq 0$ and $M = f^{-1}(0)$, then $\forall p \in M$: $T_pM = \{v \in \mathbb{R}^n : \langle \nabla f, v \rangle = 0\}$



so $\nabla f \perp T_pM$.

Proposition: If $f: U \subseteq \mathbb{R}^n \to \mathbb{R}^k$ has constant rank r and $M = f(U)$ is a submanifold of $\mathbb{R}^k$ (it has to be of dimension r)

Then, $\forall p \in U$, $T_{f(p)} M = \operatorname{Im}(df_p)$.

<u>Proof:</u> For any $v \in \mathbb{R}^n$ If $\gamma : (-\varepsilon, \varepsilon) \to U$, $\gamma(0) = p$, $\dot{\gamma}(0) = v$:

The curve $\tilde{\gamma}(t) = f(\gamma(t))$ lies in $M = f(U)$, and $\tilde{\gamma}(0) = f(p)$

So: $\dfrac{d}{dt} \tilde{\gamma}(t) \Big|_{t=0} \in T_{f(p)} M$

$\Rightarrow df_p(\underbrace{\dot{\gamma}(0)}_{= v}) \in T_{f(p)} M$. So $df_p(\mathbb{R}^n) \subseteq T_{f(p)} M$

$\Rightarrow \operatorname{Im}(df_p) \subseteq T_{f(p)} M$

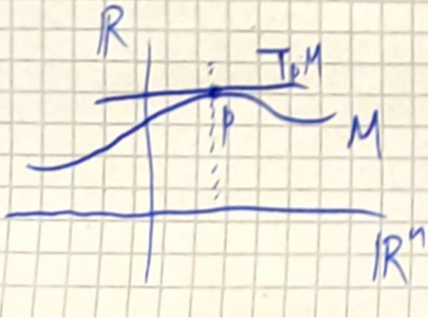
But $\dim(\operatorname{Im}(df_p)) = \operatorname{rank}_f(p) = r = \dim T_{f(p)} M$

$\Rightarrow \operatorname{Im}(df_p) = T_{f(p)} M$. ☺

<u>Remark:</u> $T_{f(p)} M$ is spanned by $\{df_p(e_1), \dots, df_p(e_n)\}$, if $\{e_1, \dots, e_n\}$ is a basis of $\mathbb{R}^n$; these are the columns of the Jacobian matrix.

<u>Example:</u>

Let $\phi : U \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function, and $M \subset \mathbb{R}^{n+1}$ be it's graph (i.e. $M = \{(x, f(x)) : x \in U \subseteq \mathbb{R}^n\}$)



Suppose that $p = (0, f(0)) \in M$.

Let's determine $T_p M$

a) Implicit point of view: M is the level set $\{f=0\}$ of $f: U\times\mathbb{R}\to\mathbb{R}$, $f(x,y) = y-\phi(x)$. So $\nabla f = (-\nabla\phi, 1)$

So $T_pM = \{ v\in\mathbb{R}^{n+1} : \langle v, \nabla f_p\rangle = 0\}$

$$= \{ (\bar v, w)\in\mathbb{R}^n\times\mathbb{R} : -\langle \bar v, \nabla\phi\rangle + w = 0\}$$

$$= \left\{ (\bar v, w)\in\mathbb{R}^n\times\mathbb{R} : w = \bar v_1 \frac{\partial_1\phi(0)}{\partial x_1} + \dots + \bar v_n \frac{\partial_n\phi}{\partial x_n}(0) \right\}$$

b) Parametric point of view:

M is the image of the map $F: U\subseteq\mathbb{R}^n\to\mathbb{R}^{n+1}$,

$F(x) = (x, \phi(x))$. It has constant rank n.

So $$DF = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial F_{n+1}}{\partial x_1} & \cdots & \frac{\partial F_{n+1}}{\partial x_1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \\ \frac{\partial\phi}{\partial x_1} & & \cdots & \frac{\partial\phi}{\partial x_n} \end{bmatrix}$$

$T_pM = \operatorname{Im}(dF_0) = \operatorname{span}\{DF_1(0), \dots, DF_n(0)\}$

$$= \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \frac{\partial\phi}{\partial x_1}(0) \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \frac{\partial\phi}{\partial x_n}(0) \end{bmatrix} \right\} \subseteq \mathbb{R}^{n+1}$$

<u>Definition:</u> Affine tangent space at $p\in M$

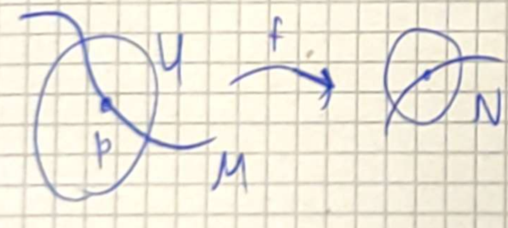
$$A_pM = T_pM + p.$$

Differentiable maps between submanifolds:

Since a submanifold $M \subseteq \mathbb{R}^n$ of dimension m locally (in appropriate curvilinear coordinates) looks like $\mathbb{R}^m$.

We can define the differentiability properties of functions on M similarly as for functions on $\mathbb{R}^m$.

Definition: Let $M \subseteq \mathbb{R}^k$ and $N \subseteq \mathbb{R}^l$ be submanifolds of class C^k. A map $f: M \to N$ is <u>differentiable of</u> class $C^{k'}$ $(k' \leq k)$ in a neighborhood of $p \in M$ if there exist an open neighb. U of p in $\mathbb{R}^k$ and a $C^{k'}$ function $F: U \to \mathbb{R}^l$ such that $F|_{M \cap U} = f|_{M \cap U}$ (i.e. F is a $C^{k'}$ extension of f).



<u>Note</u>: We cannot, in general, construct $C^{k'}$ extensions if $k' > k$.

Note:

a) $f \in C^k(M,N)$ is a <u>diffeomorphism</u> if it is <u>bijective</u> and $f^{-1} \in C^k(N,M)$

b) A <u>local parametrization</u> of M: A map $\psi: U \subseteq \mathbb{R}^m \to \psi(U) \subseteq M$ which is a diffeomorphism on its image (if $\dim M = m$).

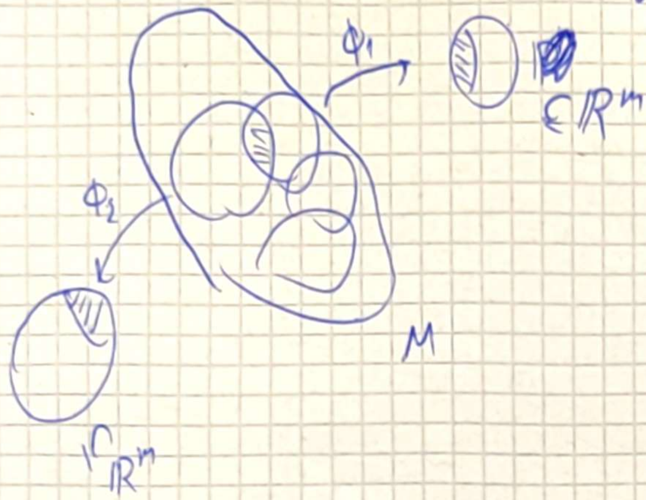
- Global parametrization: If $\psi(U) = M$

c) A <u>local chart</u>: Is the inverse of a local parametrization, i.e.

a diffeomorphism $\phi: V \subseteq M \longrightarrow \phi(V) \subseteq \mathbb{R}^m$

In general: M can be "covered" by local charts,

i.e. M can be viewed as a "gluing" of open subsets of $\mathbb{R}^m$



Note:

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow \Phi & & \downarrow \Psi \\ \mathbb{R}^m & & \mathbb{R}^n \end{array}$$

If $\Phi: U \subseteq M \to \Phi(U) \subseteq \mathbb{R}^m$

and $\Psi: V \subseteq N \to \Psi(V) \subseteq \mathbb{R}^n$

are C^k local charts, then

$f: U \to V$ is a $C^{k'}$ map if

$\Psi \circ f \circ \Phi^{-1}: \Phi(U) \subseteq \mathbb{R}^m \to \Psi(V) \subseteq \mathbb{R}^n$

is a $C^{k'}$ map from $\mathbb{R}^m$ to $\mathbb{R}^n$, $(k' \leq k)$

Also note:

If $f: M \to N$ is differentiable ~~at~~ of class C^1 around $p \in M$, then

any C^1 extension F of f on $U \ni p$

satisfies $dF_p|_{T_pM} \subseteq T_{f(p)}N$, and the restriction map

$dF_p|_{T_pM}: T_pM \longrightarrow T_{f(p)}N$ doesn't depend on the extension F

(so $df_p: T_pM \to T_{f(p)}N$ is well-defined).

Proof: Let $v \in T_pM$. Then $\exists$ curve $\gamma: (-\varepsilon, \varepsilon) \to M \cap U$ with $\gamma(0) = p$, $\dot\gamma(0) = v$, and

$$dF_p(v) = dF_p(\dot\gamma(0)) = \frac{d}{dt}\left(F \circ \gamma(t)\right)\Big|_{t=0} \overset{\gamma(t) \in M \cap U}{=} \frac{d}{dt}\left(\underbrace{f \circ \gamma(t)}_{\in N}\right)\Big|_{t=0}$$

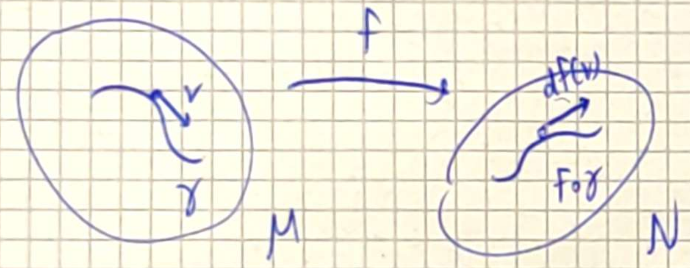
so it doesn't depend on the extension F.

so this belongs to $T_{f(p)}N$.

Also this map is clearly linear in $v = \dot\gamma(0)$.

$\blacksquare$

So:



So df: maps the velocity vector of γ to that of $f \circ \gamma$.

Let $M \subset \mathbb{R}^n$ be a C^k submanifold of dimension m.

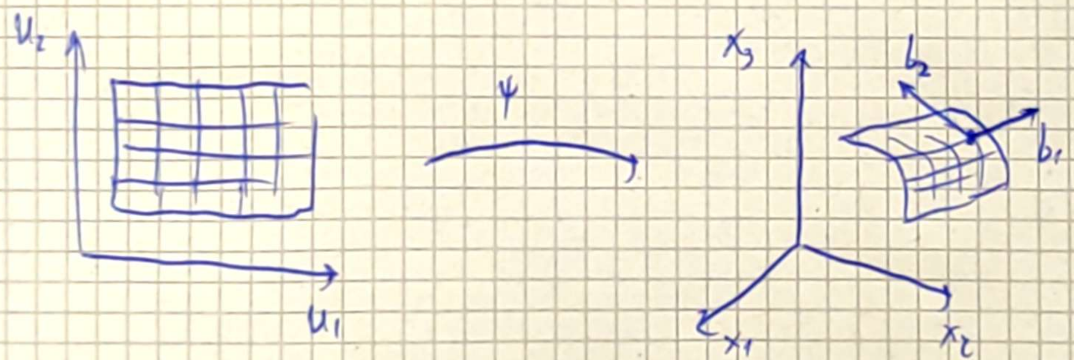
Let $U \subseteq \mathbb{R}^m$ be open and $\psi: U \longrightarrow M$ a local parametrization of M of class C^k.

• ψ, as a map from $U \subseteq \mathbb{R}^m$ to $\mathbb{R}^n$ must have full rank ($=m$).

So if $(u_1, \dots, u_m)$ are coordinates on U, then, the vectors

$$\vec{b}_1 = \frac{\partial \vec{\psi}}{\partial u_1}, \dots, \vec{b}_m = \frac{\partial \vec{\psi}}{\partial u_m}$$

are linearly independent vectors of $\mathbb{R}^n$ spanning $T_{\psi(u)}M$.



We usually choose Cartesian coordinates on U.

Definition: • The functions $u_1, \dots, u_m : U \to \mathbb{R}$; local coordinates on M associated to the parametrization ψ (they can be viewed as functions on M : $u_1 \circ \psi^{-1}, \dots, u_m \circ \psi^{-1} : \psi(U) \subseteq M \longrightarrow \mathbb{R}$)

• The vectors $b_1, \dots, b_m$: The corresponding coordinate basis of $T_{\psi(u)}M$.